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ON THE APPLICATION OF BIMETRIC RELATIONS IN ELASTICITY

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Abstract—The compatibility conditions of elasticity are given a bimetric formulation, based upon a systematic use of covariant differentiation with respect to the reference configuration. Both three-dimensional classical elasticity and Cosserat two-dimensional surfaces are considered.

1. INTRODUCTION

The purpose of this paper is to apply the bimetric formulae of differential geometry to elasticity.

The bimetric standpoint was first introduced by Rosen (1980) in relativistic physics. It represents the basis on which several alternative theories of gravitational field are founded, but is not necessarily related to general relativistic universes only. We believe it may be possible to apply a bimetric approach to some basic formulae of nonrelativistic continuum theory.

Elaborating those questions we established, in particular, that the bimetric approach is more direct and that it simplifies the procedure leading to the compatibility conditions. Also, in the bimetric formulation, the geometric picture of these conditions seems clearer to us.

2. BIMETRIC RELATIONS

We shall give a brief account of the basic formulae characterizing the bimetric approach to classical differential geometry.

If a three-dimensional metric, defined by the metric tensor $g_{\mu\rho}(X^{\nu})$, $(\mu, \rho, \nu = 1, 2, 3)$, is obtained by the deformation of another metric $\gamma_{\mu\rho}$, expressed with respect to the same coordinates X^{ν}, the following relations hold (Rosen, 1980):

$$\begin{Bmatrix} \pi \\ \mu\nu \end{Bmatrix} = \Gamma^{\pi}_{\mu\nu} + \Delta^{\pi}_{.\mu\nu} \tag{1}$$

$$R^{\lambda}_{.\mu\nu\rho} = P^{\lambda}_{.\mu\nu\rho} + K^{\lambda}_{.\mu\nu\rho} \tag{2}$$

where

$$\begin{Bmatrix} \pi \\ \mu\nu \end{Bmatrix} = \tfrac{1}{2} g^{\pi\tau}\left(\frac{\partial g_{\tau\nu}}{\partial X^{\mu}} + \frac{\partial g_{\mu\tau}}{\partial X^{\nu}} - \frac{\partial g_{\mu\nu}}{\partial X^{\tau}}\right),$$

$$\Gamma^{\pi}_{\mu\nu} = \tfrac{1}{2} \gamma^{\pi\tau}\left(\frac{\partial \gamma_{\tau\nu}}{\partial X^{\mu}} + \frac{\partial \gamma_{\mu\tau}}{\partial X^{\nu}} - \frac{\partial \gamma_{\mu\nu}}{\partial X^{\tau}}\right),$$

$$\Delta^{\pi}_{.\mu\nu} = \tfrac{1}{2} g^{\pi\tau}(g_{\tau\nu|\mu} + g_{\mu\tau|\nu} - g_{\mu\nu|\tau}). \tag{3}$$

$R^{\lambda}_{.\mu\nu\rho}$ and $P^{\lambda}_{.\mu\nu\rho}$ are the Riemann–Christoffel tensors corresponding to the metrics $g_{\mu\rho}$ and $\gamma_{\mu\rho}$, respectively; $K^{\lambda}_{.\mu\nu\rho}$ reads:

$$K^{\lambda}_{.\mu\nu\rho} = \Delta^{\lambda}_{.\mu\rho|\nu} - \Delta^{\lambda}_{.\mu\nu|\rho} + \Delta^{\pi}_{.\mu\rho}\Delta^{\lambda}_{.\pi\nu} - \Delta^{\pi}_{.\mu\nu}\Delta^{\lambda}_{.\pi\rho}. \tag{4}$$

In the above and following formulae, a bar (|) denotes covariant derivatives with respect to the undeformed metric $\gamma_{\mu\rho}$, ∂ denotes a partial derivative, whereas a semicolon (;) denotes a covariant derivatives with respect to the deformed metric $g_{\mu\rho}$. Formulae (1)–(4) can be directly verified. In accordance with the notation chosen one has

$$g_{\mu\rho;\nu} = 0, \; \gamma_{\mu\rho|\nu} = 0; \quad g_{\mu\rho|\nu} \neq 0, \; \gamma_{\mu\rho;\nu} \neq 0. \tag{5}$$

3. DEFORMABLE BODIES: COMPATIBILITY CONDITIONS

In the classical mechanics of continua the undeformed (reference) configuration of a body $\mathscr{B}$, $K_0(X^K)$, $K = 1, 2, 3$, is the set of points of that body which occupy, at the initial moment t_0, a domain V in E_3, bounded by a closed surface $\mathscr{A}$. The position vector of a particle is $\mathbf{R}(X^K)$, where X^K are the material (Lagrangian) coordinates, $K = 1, 2, 3$. The deformed configuration $K(X^K, t)$ is the set of points of a body $\mathscr{B}$ which occupy, at a moment t, a domain v in E_3, bounded by a closed surface a. The position vector of a particle in that configuration is $\mathbf{r}(x^k)$, where x^k are the spatial (Eulerian) coordinates, $k = 1, 2, 3$.

The square of a line element in the undeformed and deformed configurations are, respectively,

$$dS^2 = G_{KL}(X^M)\, dX^K\, dX^L = c_{kl}(x^m, t)\, dx^k\, dx^l, \tag{6}$$

$$ds^2 = C_{KL}(X^M, t)\, dX^K\, dX^L = q_{kl}(x^m)\, dx^k\, dx^l, \tag{7}$$

where G_{KL} is the metric tensor in K_0 and C_{KL} is the Green deformation tensor, which can be interpreted as the metric tensor of the deformed configuration if one considers X^K as convective coordinates. The tensor g_{kl} is the fundamental metric tensor of the deformed configuration K and c_{kl} is the Cauchy deformation tensor.

It is convenient to take, instead of C_{KL} and c_{kl}, the strain measures E_{KL} and c_{kl}, where

$$2E_{KL} = C_{KL} - G_{KL}, \quad 2c_{kl} = q_{kl} - c_{kl}. \tag{8}$$

Considering the undeformed configuration K_0 as an undeformed metric with the fundamental tensor G_{KL} and the deformed configuration as a deformed metric with the fundamental tensor C_{KL}, there results from (5):

$$G_{KL|M} = 0, \quad C_{KL;M} = 0; \quad G_{KL;M} \neq 0, \quad C_{KL|M} \neq 0, \tag{9}$$

(1) and (2) now read

$$\begin{Bmatrix} K \\ LM \end{Bmatrix} = \Gamma^K_{LM} + \Delta^K_{.LM} \tag{10}$$

$$R^L_{.MNR} = P^L_{.MNR} + K^L_{.MNR}, \tag{11}$$

where we have correspondingly for (3):

$$\begin{Bmatrix} K \\ LM \end{Bmatrix} = \tfrac{1}{2}C^{KP}\left(\frac{\partial C_{PM}}{\partial X^L} + \frac{\partial C_{LP}}{\partial X^M} - \frac{\partial C_{LM}}{\partial X^P}\right), \quad (C^{KT}C_{TS} = \delta^K_S)$$

$$\Gamma^K_{LM} = \tfrac{1}{2}G^{KP}\left(\frac{\partial G_{PM}}{\partial X^L} + \frac{\partial G_{LP}}{\partial X^M} - \frac{\partial G_{LM}}{\partial X^P}\right),$$

$$\Delta^K_{LM} = \tfrac{1}{2}C^{KP}(C_{PM|L} + C_{LP|M} - C_{LM|P}). \tag{12}$$

$R^L_{.MNR}$ and $P^L_{.MNR}$ are the Riemann–Christoffel tensors of the metrics C_{KL} and G_{KL}, respectively, and

$$K^L_{.MNR} = \Delta^L_{.MR|N} - \Delta^L_{.MN|R} + \Delta^S_{.MR}\Delta^L_{.SN} - \Delta^S_{.MN}\cdot\Delta^L_{.SR}. \tag{13}$$

Further, by (1), one has from $(9)_2$:

$$C_{KL|M} = C_{SL}\Delta^S_{.KM} + C_{KS}\Delta^S_{.LM}, \quad C^{KL}_{|M} = -C^{SL}\Delta^K_{.SM} - C^{KS}\Delta^L_{.MS}, \tag{14}$$

and

$$C_{MS|RN} - C_{MS|NR} = C_{SQ}P^Q_{MRN} + C_{MQ}P^Q_{.SRN}. \tag{15}$$

By $(12)_3$ it is easy to see that (13) can be expressed in the following form

$$K^L_{.MNR} = \tfrac{1}{2}C^{LT}(C_{TR|MN} + C_{MN|TR} - C_{RM||TN} - C_{TN|RM}) + \tfrac{1}{2}C^{LT}(C_{ST}P^S_{.MNR} + C_{MS}P^S_{.TRN}) + C^{LS}C_{QT}(\Delta^T_{.RS}\Delta^Q_{.MN} - \Delta^T_{.NS}\Delta^Q_{.MR}). \tag{16}$$

The above formula has been simplified by using the Ricci identity.

From the geometrical standpoint the compatibility conditions in the mechanics of deformable continua are connected with the metric properties of the tensor C_{KL}. For a compatible deformation, the Riemann–Christoffel tensor corresponding to the tensor C_{KL} is equal to zero, i.e. C_{KL} is the metric tensor of E_3. As $P^L_{.MNR}$ is be definition null, the compatibility conditions can, by (11), be written in the form

$$K^L_{.MNR} = 0, \tag{17}$$

which can explicitly be expressed by (8), $(9)_1$ and (16); that is,

$$E_{TR|MN} + E_{MN|TR} - E_{RM|TN} - E_{TN|RM} + C^{SQ}[(E_{SR|T} + E_{TS|R} - E_{RT|S})(E_{QM|N} + E_{NQ|M} - E_{MN|Q}) - (E_{SN|T} + E_{TS|N} - E_{NT|S})(E_{QM|R} + E_{RQ|M} - E_{RM|Q})] = 0. \tag{18}$$

This form is identical to the well-known form of the compatibility conditions for finite deformation.

In the case of infinitesimal deformations the above relations become

$$E_{TR|MN} + E_{MN|TR} - E_{RM|TN} - E_{TN|RM} = 0. \tag{19}$$

It is easy to show that one may obtain, by the use of bimetric formulae, the compatibility conditions expressed through the Cauchy deformation tensor c_{kl}, or the corresponding strain measure c_{kl}.

Application of bimetric formulae allows one to obtain the compatibility conditions of the generalized Cosserat continuum, using an appropriate expression for the line element. In this case the compatibility conditions are of the form

$$K^L_{.MNR} = 0, \quad L, M, N, R = 1, 2, \ldots, 6, \tag{20}$$

which is of the same form as in Eringen (1969).

We point out that formula (20) has been obtained only as a bimetric expression of Eringen's results. Work on Cosserat's surfaces in the next section will be based on only the general formulae of Section 2; this being the consequence of the fact that the differential geometry of surfaces in Cosserat's theory is essentially intrinsic.

4. COSSERAT SURFACES. COMPATIBILITY CONDITIONS

A Cosserat surface is a two-dimensional material surface in E_3, with a single deformable director assigned to every point of the surface.

At the initial moment t_0, the Cosserat surface $\mathscr{C}$ is in its undeformed, reference configuration. We denote this reference surface by $\mathscr{S}$, its position vector with respect to a fixed origin by $\mathbf{R}(\theta^\beta)$, $(\alpha, \beta = 1, 2)$, the assigned director by $\mathbf{D}(\theta^\beta)$, the base vectors by $\mathbf{A}_\alpha(\theta^\beta)$, the unit normal to $\mathscr{S}$ by $\mathbf{A}_\beta(\theta^\beta)$; θ^β being the convective coordinates.

At the moment t, the surface $\mathscr{C}$ is in a deformed configuration. Let us denote that surface by s, it position vector by $\mathbf{r}(\theta^\beta, t)$, the assigned director by $\mathbf{d}(\theta^\beta, t)$, the base vectors by $\mathbf{a}_\alpha(\theta^\beta, t)$, and the unit normal to s by $\mathbf{a}_\beta(\theta^\beta, t)$.

Tensors $A_{\alpha\beta}$ and $a_{\alpha\beta}$ are the first metric tensor of the surface $\mathscr{S}$ and s, and tensors $B_{\alpha\beta}$ and $b_{\alpha\beta}$ the second, and so (Naghdi, 1972),

$$A_{\alpha\beta} = \mathbf{A}_\alpha \cdot \mathbf{A}_\beta, \quad B_{\alpha\beta} = -\mathbf{A}_\alpha \frac{\partial \mathbf{A}_3}{\partial \theta^\beta} = \mathbf{A}_3 \cdot \frac{\partial \mathbf{A}_\alpha}{\partial \theta^\beta}, \quad \mathbf{A}_\alpha = \frac{\partial \mathbf{R}}{\partial \theta^\alpha}, \quad \mathbf{A}_\alpha \cdot \mathbf{A}_3 = 0,$$

$$a_{\alpha\beta} = \mathbf{a}_\alpha \cdot \mathbf{a}_\beta, \quad b_{\alpha\beta} = -\mathbf{a}_\alpha \cdot \frac{\partial \mathbf{a}_3}{\partial \theta^\beta} = \mathbf{a}_3 \cdot \frac{\partial \mathbf{a}_\alpha}{\partial \theta^\beta}, \quad \boldsymbol{\alpha}_\alpha = \frac{\partial \mathbf{r}}{\partial \theta^\alpha}, \quad \mathbf{a}_\alpha \cdot \boldsymbol{\alpha}_3 = 0. \tag{21}$$

The directors $\mathbf{D}$ and $\mathbf{d}$ and the gradient directors $\mathbf{D}_{,\alpha}$ and $\mathbf{d}_{,\alpha}$ can be expressed in the form (Green *et al.*, 1965),

$$\mathbf{D} = D_\alpha \mathbf{A}^\alpha + D_3 \mathbf{A}^3, \quad \mathbf{d} = d_\alpha \mathbf{a}^\alpha + d_3 \mathbf{a}^3, \quad \left(\mathbf{D}_{,\alpha} = \frac{\partial \mathbf{D}}{\partial \theta^\alpha}\right)$$

$$\mathbf{D}_{,\alpha} = \Lambda_{\beta\alpha} \mathbf{A}^\alpha + \Lambda_{3\alpha} \mathbf{A}^3, \quad \mathbf{d}_{,\alpha} = \lambda_{\beta\alpha} \mathbf{a}^\alpha + \lambda_{3\alpha} \mathbf{a}^3, \quad \left(\mathbf{d}_{,\alpha} = \frac{\partial \mathbf{d}}{\partial \theta^\alpha}\right) \tag{22}$$

where $\mathbf{A}^\alpha$ and $\mathbf{a}^\alpha$ are reciprocal basic vectors and $\mathbf{A}_3 = \mathbf{A}^3$, $\mathbf{a}_3 = \mathbf{a}^3$. Further,

$$\Lambda_{\beta\alpha} = \mathbf{D}_{,\alpha} \cdot \mathbf{A}_\beta, \; \Lambda_{3\alpha} = \mathbf{D}_{,\alpha} \cdot \mathbf{A}_3; \quad \lambda_{\beta\alpha} = \mathbf{d}_{,\alpha} \cdot \mathbf{a}_\beta, \; \lambda_{3\alpha} = \mathbf{d}_{,\alpha} \cdot \mathbf{a}_3 \tag{23}$$

$$\Lambda_{\beta\alpha} = D_{\beta|\alpha} - B_{\alpha\beta} D_3, \; \Lambda_{3\alpha} = D_{3|\alpha} + B^\gamma_\alpha D_\gamma; \quad \lambda_{\beta\alpha} = d_{\beta;\alpha} - b_{\alpha\beta} d_3, \; \lambda_{3\alpha} = d_{3;\alpha} + b^\gamma_\alpha d_\gamma. \tag{24}$$

The quantities $A_{\alpha\beta}$, $\Lambda_{\alpha\beta}$, $\Lambda_{3\alpha}$, D_α and D_3, respectively $a_{\alpha\beta}$, $\lambda_{\alpha\beta}$, $\lambda_{3\alpha}$, d_α and d_3, are the kinematical strain measures which completely determine the Cosserat surface in the undeformed, respectively in the deformed, configuration (Green *et al.*, 1965). One may use, instead, more suitable strain measures of the form

$$\begin{aligned} c_{\alpha\beta} &= \tfrac{1}{2}(a_{\alpha\beta} - A_{\alpha\beta}), \quad \kappa_{\alpha\beta} = \lambda_{\alpha\beta} - \Lambda_{\alpha\beta}, \quad \gamma_\alpha = d_\alpha - D_\alpha \\ \kappa_{3\alpha} &= \lambda_{3\alpha} - \Lambda_{3\alpha}, \quad \gamma_3 = d_3 - D_3. \end{aligned} \tag{25}$$

Let us remark that the quantity $\eta_{\alpha\beta}$, defined by

$$\eta_{\alpha\beta} = b_{\alpha\beta} - B_{\alpha\beta}, \tag{26}$$

can be equivalently introduced when considering deformations. These strain measures are not independent of measures (25), namely, from $(24)_{1,3}$, $(25)_{2,3,5}$ and (26), we obtain

$$\kappa_{\alpha\beta} = \gamma_{\alpha|\beta} - (D_\pi + \gamma_\pi)\Delta^\pi_{.\alpha\beta} - B_{\alpha\beta}\gamma_3 - \eta_{\alpha\beta}(D_3 + \gamma_3) \tag{27}$$

where we have, with respect to (1), made use of the relation

$$d_{\alpha;\beta} = d_{\alpha|\beta} - d_{\pi}\Delta^{\pi}_{.\alpha\beta}, \tag{28}$$

whereas one obtains from $(24)_{2,4}$, $(25)_{3,4,5}$ and (26),

$$\kappa_{3\alpha} = \gamma_{3|\alpha} + B^{\nu}_{\alpha}\gamma_{\nu} + \eta^{\nu}_{\alpha}(D_{\nu} + \eta^{\nu}_{\alpha}(D_{\nu} + \gamma_{\nu}) \quad \left(d_{3;\alpha} = d_{3|\alpha} = \frac{\partial d_3}{\partial \theta^{\alpha}}\right). \tag{29}$$

If we consider the undeformed configuration of $\mathscr{C}$ as an undeformed metric space and its deformed configuration as a deformed metric space, there results by (5)

$$a_{\alpha\beta;\gamma} = 0,\ A_{\alpha\beta|\gamma} = 0; \quad a_{\alpha\beta|\gamma} \neq 0,\ A_{\alpha\beta;\gamma} \neq 0. \tag{30}$$

one has by (1)

$$\begin{Bmatrix} \pi \\ \mu\nu \end{Bmatrix} = \tfrac{1}{2}a^{\pi\tau}\left(\frac{\partial a_{\tau\nu}}{\partial \theta^{\mu}} + \frac{\partial a_{\mu\tau}}{\partial \theta^{\nu}} - \frac{\partial a_{\mu\nu}}{\partial \theta^{\tau}}\right),$$

$$\Gamma^{\pi}_{\mu\nu} = \tfrac{1}{2}A^{\pi\tau}\left(\frac{\partial A_{\tau\nu}}{\partial \theta^{\mu}} + \frac{\partial A_{\mu\tau}}{\partial \theta^{\nu}} - \frac{\partial A_{\mu\nu}}{\partial \theta^{\tau}}\right),$$

$$\Delta^{\pi}_{.\mu\nu} = \tfrac{1}{2}a^{\pi\tau}(a_{\tau\nu|\mu} + a_{\mu\tau|\nu} - \alpha_{\mu\nu|\tau}), \tag{31}$$

and $(31)_3$ can, by $(25)_1$ and $(30)_2$, be expressed in the form

$$\Delta^{\pi}_{.\mu\nu} = a^{\pi\tau}(e_{\tau\nu|\mu} + e_{\mu\tau|\nu} - e_{\mu\nu|\tau}). \tag{32}$$

The relations between the covariant derivatives of the second-order tensors read, by (1) and (31), for the deformed an undeformed metrics, respectively,

$$\begin{aligned} t_{\alpha\beta;\gamma} &= t_{\alpha\beta|\gamma} - t_{\pi\beta}\Delta^{\pi}_{.\alpha\gamma} - t_{\alpha\pi}\Delta^{\pi}_{.\beta\gamma} \\ t^{\alpha\beta}_{;\gamma} &= t^{\alpha\beta}_{|\gamma} + t^{\pi\beta}\Delta^{\alpha}_{.\pi\gamma} + t^{\alpha\pi}\Delta^{\beta}_{.\pi\gamma} \\ t^{\alpha}_{.\beta;\gamma} &= t^{\alpha}_{.\beta|\gamma} + t^{\pi}_{.\beta}\Delta^{\alpha}_{.\pi\gamma} - t^{\alpha}_{.\pi}\Delta^{\pi}_{.\beta\gamma}. \end{aligned} \tag{33}$$

The compatibility conditions for deformations in the Cosserat theory are the conditions of integrability of the partial differential equations

$$\begin{aligned} &\frac{\partial \mathbf{A}_{\alpha}}{\partial \theta^{\beta}} = \Gamma^{\nu}_{\alpha\beta}\mathbf{A}_{\nu} + B_{\alpha\beta}\mathbf{A}_3, \quad \frac{\partial \mathbf{D}}{\partial \theta^{\alpha}} = \Lambda_{\beta\alpha}\mathbf{A}^{\beta} + \Lambda_{3\alpha}\mathbf{A}^3 \\ &\frac{\partial \mathbf{a}_{\alpha}}{\partial \theta^{\beta}} = \begin{Bmatrix} \nu \\ \alpha\beta \end{Bmatrix}\mathbf{a}_{\nu} + b_{\alpha\beta}\mathbf{a}_3, \quad \frac{\partial \mathbf{d}}{\partial \theta^{\alpha}} = \lambda_{\beta\alpha}\mathbf{a}^{\beta} + \lambda_{3\alpha}\mathbf{a}^3, \end{aligned} \tag{34}$$

which can be expressed in the form

$$\begin{aligned} P_{\alpha\beta\gamma\delta} &= B_{\alpha\gamma}B_{\beta\delta} - B_{\alpha\delta}B_{\beta\gamma}, \quad B_{\nu\alpha|\beta} = B_{\nu\beta|\alpha}, \\ \Lambda_{\nu\alpha|\beta} + B_{\nu\alpha}\Lambda_{3\beta} &= \Lambda_{\nu\beta|\alpha} + B_{\nu\beta}\Lambda_{3\alpha}, \\ \Lambda_{3\alpha|\beta} + B^{\nu}_{\beta}\Lambda_{\nu\alpha} &= \Lambda_{3\beta|\alpha} + B^{\nu}_{\alpha}\Lambda_{\nu\beta}, \end{aligned} \tag{35}$$

on the undeformed surface $\mathscr{S}$, and in the form

$$\begin{aligned} R_{\alpha\beta\gamma\delta} &= b_{\alpha\gamma}b_{\beta\delta} - b_{\alpha\delta}b_{\beta\gamma}, \quad b_{\nu\alpha;\beta} = b_{\nu\beta;\alpha}, \\ \lambda_{\nu\alpha;\beta} + b_{\nu\alpha}\lambda_{3\beta} &= \lambda_{\nu\beta;\alpha} + b_{\nu\beta}\lambda_{3\alpha}, \\ \lambda_{3\alpha;\beta} + B^{\nu}_{\beta}\lambda_{\nu\alpha} &= \lambda_{3\beta;\alpha} + B^{\nu}_{\alpha}\lambda_{\nu\beta}, \end{aligned} \tag{36}$$

on the deformed surface s. $P_{\alpha\beta\gamma\delta}$ and $R_{\alpha\beta\gamma\delta}$ are the Riemann–Christoffel tensors of $\mathscr{S}$ and s, respectively. The first two eqns in (35) and (36) are the well-known Gauss and Codazzi equations. It is clear, by (24), that (35) and (36) may be expressed in another form. Meanwhile, if applying the bimetric relations to the compatibility conditions of Cosserat surfaces through the deformation measures (25), we can retain the form of (35) and (36) for these conditions.

The first of eqns (36), expressed in terms of the strain measures $(25)_1$ and (26), will read, by the bimetric relation (2),

$$R_{\tau\mu\nu\rho} = a_{\lambda\tau} P^{\lambda}_{.\mu\nu\rho} + a_{\lambda\tau} K^{\lambda}_{.\mu\nu\rho} \tag{37}$$

where

$$K^{\lambda}_{.\mu\nu\rho} = \tfrac{1}{2} a^{\lambda\gamma}(a_{\gamma\rho|\mu\nu} + a_{\mu\nu|\gamma\rho} - a_{\rho\mu|\gamma\nu} - a_{\gamma\nu|\rho\mu}) + \tfrac{1}{2} a^{\lambda\gamma}(a_{\pi\gamma} P^{\pi}_{.\mu\rho\nu} + a_{\mu\pi} P^{\pi}_{.\gamma\rho\nu}) + a^{\lambda\pi} a_{\delta\gamma}(\Delta^{\gamma}_{.\rho\pi}\Delta^{\delta}_{.\mu\nu} - \Delta^{\gamma}_{.\nu\pi}\Delta^{\delta}_{.\mu\rho}). \tag{38}$$

We obtained (38) from (4), by the use of $(30)_1$, $(31)_3$, $(33)_3$ and the following relation:

$$a_{\mu\gamma|\rho\nu} - a_{\mu\gamma|\nu\rho} = a_{\pi\gamma} P^{\pi}_{.\mu\rho\nu} + a_{\mu\pi} P^{\pi}_{.\gamma\rho\nu}. \tag{39}$$

There results from (32), $(35)_1$, $(36)_1$, (37) and (38), by $(25)_1$, (26) and $(30)_2$

$$\varepsilon^{\alpha\beta}\varepsilon^{\delta\gamma}[e_{\alpha\delta|\beta\gamma} + B_{\beta\gamma}(\eta_{\alpha\delta} - B^{\lambda}_{\delta} e_{\lambda\alpha})] + \varepsilon^{\delta\gamma}\eta_{\alpha\delta}\eta_{\beta\gamma} + \varepsilon^{\delta\gamma} a^{\rho\mu}(e_{\rho\delta|\alpha} + e_{\alpha\rho|\delta} - e_{\alpha\delta|\rho})(e_{\mu\gamma|\beta} + e_{\beta\mu|\gamma} - e_{\beta\gamma|\mu}) = 0 \tag{40}$$

where $\varepsilon^{\pi\nu}$ is the Ricci skew-symmetric tensor.

By $(25)_1$, (26), $(31)_3$ and $(35)_2$, respectively (32), we transform $(36)_2$ into the form

$$\varepsilon^{\alpha\beta}[\eta_{\nu\alpha|\beta} - a^{\pi\sigma}(B_{\pi\alpha} + \eta_{\pi\alpha})(e_{\sigma\beta|\nu} + e_{\nu\sigma|\beta} - e_{\nu\beta|\sigma})] = 0. \tag{41}$$

By $(25)_{1,2,4}$, (26), $(31)_3$ and $(35)_2$, respectively (32), we transform $(36)_3$ into the form

$$\varepsilon^{\alpha\beta}[\chi_{\nu\alpha|\beta} + \alpha^{\pi\tau}(\Lambda_{\pi\beta} + \kappa_{\pi\beta})(e_{\tau\nu|\alpha} + e_{\alpha\tau|\nu} - e_{\alpha\nu|\tau}) + B_{\nu\alpha}\kappa_{3\beta} + \eta_{\nu\beta}(\Lambda_{3\alpha} + \kappa_{3\alpha})] = 0. \tag{42}$$

Finally, $(36)_4$ is, by $(25)_{2,4}$, (26) and $(35)_4$, transformed into the form

$$\varepsilon^{\alpha\beta}[\kappa_{3\alpha|\beta} + B^{\nu}_{\beta}\kappa_{\nu\alpha} + \eta^{\nu}_{\beta}(\Lambda_{\nu\alpha} + \kappa_{\nu\alpha})] = 0, \tag{43}$$

where we have made use of

$$\kappa_{3\alpha;\beta} = \kappa_{3\alpha|\beta} - \kappa_{3\pi}\Delta^{\pi}_{.\alpha\beta}, \tag{44}$$

which results from the application of (1) in $\kappa_{3\alpha;\beta}$.

Relations (40)–(43) represent compatibility conditions for the deformation of the Cosserat surface. They have been obtained from the general integrability conditions (34), and by the use of bimetric relations (1) and (2).

It is easy to obtain from (40)–(43) the compatibility conditions for deformation in some special cases, like for instance, the restricted theory (Naghdi, 1972), or the case of a flat undeformed configuration. For some special positions of the initial director these relations are simplified.

In the restricted theory, i.e. in the theory in which the director is not admitted, the deformation is determined only by strain measures $e_{\alpha\beta}$ and $\eta_{\alpha\beta}$, and we have

$$\mathbf{D} \equiv \mathbf{A}_3, \quad \mathbf{d} \equiv \mathbf{a}_3, \quad \Lambda_{\alpha\beta} = -B_{\alpha\beta}, \quad \Lambda_{3\alpha} = 0, \quad \kappa_{\alpha\beta} = -\eta_{\alpha\beta}, \quad \kappa_{3\alpha} = 0. \tag{45}$$

Equations (40) and (41) retain their form, eqn (42) reduces to (41), and (43) is identically satisfied.

If the undeformed space is flat there follows $B_{\alpha\beta} = 0$, so that by (40)–(43) one obtains the corresponding compatibility conditions. Also, if one chooses $\mathbf{D} \equiv \mathbf{A}_3$ one has

$$D_\alpha = 0, \quad \Lambda_{\alpha\beta} = -B_{\alpha\beta}, \quad \Lambda_{3\alpha} = 0, \tag{46}$$

so that eqns (40)–(43) take a simpler form.

5. INFINITESIMAL DEFORMATIONS. COMPATIBILITY CONDITIONS FOR SHELLS AND PLATES

The results of the linear theory of Cosserat surfaces provide the basis of a linear kinematical theory of shells and plates by a direct approach. As a consequence, the compatibility conditions (40)–(43) for infinitesimal deformations hold for the deformations in the linear theory of shells and plates.

Assuming the deformations infinitesimal, we have, first, for the contravariant metric tensor of the deformed space,

$$a^{\alpha\beta} = A^{\alpha\beta} - 2e^{\alpha\beta}. \tag{47}$$

Neglecting in (40)–(43) terms of higher order, we obtain the compatibility conditions of the Cosserat surface in the case of infinitesimal deformations, respectively the compatibility conditions of the linear theory of shells and plates in the direct approach, of the form

$$\varepsilon^{\alpha\beta}\varepsilon^{\delta\gamma}[e_{\alpha\delta|\beta\gamma} + B_{\beta\gamma}(\eta_{\alpha\delta} - B^{\sigma}_{\delta}e_{\sigma\alpha})] = 0, \tag{48}$$

$$\varepsilon^{\alpha\beta}[\eta_{\nu\alpha|\beta} - B^{\sigma}_{\alpha}(e_{\sigma\beta|\nu} + e_{\sigma\nu|\beta} - e_{\nu\beta|\sigma})] = 0, \tag{49}$$

$$\varepsilon^{\alpha\beta}[\kappa_{\nu\alpha|\beta} + \Lambda^{\sigma}_{\beta}(e_{\sigma\nu|\alpha} + e_{\alpha\sigma|\nu} - e_{\alpha\nu|\sigma}) + B_{\nu\alpha}\kappa_{3\beta} + \Lambda_{3\alpha}\eta_{\nu\beta}] = 0, \tag{50}$$

$$\varepsilon^{\alpha\beta}(\kappa_{3\alpha|\beta} + B^{\nu}_{\beta}\kappa_{\nu\alpha} + \Lambda_{\nu\alpha}\eta^{\nu}_{\beta}) = 0. \tag{51}$$

Finally, we shall compare eqns (48)–(51) with the corresponding equations quoted in Naghdi (1972), which represent the compatibility conditions of the linear theory of shells and plates in the case $\mathbf{D} \equiv \mathbf{A}_3$. By (25) and (46), expressions (27) and (29) reduce to

$$\kappa_{\alpha\beta} = \gamma_{\alpha|\beta} - B_{\alpha\beta}\gamma_3 - \eta_{\alpha\beta} \tag{52}$$

$$\kappa_{3\alpha} = \gamma_{3|\alpha} + B^{\sigma}_{\alpha}\gamma_{\sigma}, \tag{53}$$

whereas there results, from (48) using (52),

$$\varepsilon^{\alpha\beta}\varepsilon^{\delta\gamma}[e_{\alpha\delta|\gamma\beta} + B_{\beta\gamma}(\gamma_{\alpha|\delta} - B_{\alpha\delta}\gamma_3 - \kappa_{\alpha\delta} - B^{\sigma}_{\delta}e_{\sigma\alpha})] = 0. \tag{54}$$

$\eta_{\alpha\delta}$ being symmetric, one obtains by (52)

$$\gamma_{\alpha|\delta} - \kappa_{\alpha\delta} = \gamma_{\delta|\alpha} - \kappa_{\delta\alpha}. \tag{55}$$

Then (54) reduces to

$$\varepsilon^{\alpha\beta}\varepsilon^{\delta\gamma}[e_{\alpha\delta|\gamma\beta} - B_{\beta\gamma}(\kappa_{\delta\alpha} + B^{\sigma}_{\alpha}e_{\sigma\delta} - \gamma_{\delta|\alpha} + B_{\alpha\delta}\gamma_3)] = 0. \tag{56}$$

Further, (49) can be written in the equivalent form

$$\varepsilon^{\alpha\beta}[\varepsilon^{\nu\gamma}(\eta_{\nu\alpha|\beta} - B^{\sigma}_{\alpha}e_{\nu\sigma|\beta}) - \varepsilon^{\nu\sigma}B^{\gamma}_{\nu}e_{\sigma\alpha|\beta}] = 0, \tag{57}$$

wherefrom, by (52),

$$\varepsilon^{\alpha\beta}[\varepsilon^{\nu\gamma}(\kappa_{\nu\alpha|\beta} + B^{\sigma}_{\alpha}e_{\nu\sigma|\beta} - \gamma_{\nu|\alpha\beta} + B_{\nu\alpha}\gamma_{3|\beta}) + \varepsilon^{\nu\sigma}B^{\gamma}_{\nu}e_{\sigma\alpha|\beta}] = 0. \tag{58}$$

According to (52), (53) and (49), one obtains from (50)

$$\varepsilon^{\alpha\beta}(\gamma_{\nu|\alpha\beta} + B_{\nu\alpha}B^{\sigma}_{\beta}\gamma_{\sigma}) = 0, \tag{59}$$

and from (51), by (52) and (53),

$$\varepsilon^{\alpha\beta}(\kappa_{3\alpha|\beta} - B^{\sigma}_{\alpha}\gamma_{\sigma|\beta}) = 0. \tag{60}$$

Equations (56), (58), (59) and (60) have the same form as the corresponding relations in Naghdi (1972), which have been obtained as the necessary and sufficient conditions for the existence of single-valued fields of displacements and director displacements in the case of infinitesimal deformations.

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